Is the Deformation Parameter in q-Rotor Model Really Phenomenological ?

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Abstract

We cast the q-rotor in the framework of Barnett-Pegg theory for rotation angle, whose underlying algebra is $SU_q(2)$. A new method to fix the deformation parameter from the theory is suggested. We test our ideas by fitting rotational spectra in deformed even-even and superdeformed nuclei. The results are in good agreement with the previous phenomenological applications of q-rotor model.

In recent years, quantum algebras have been extensively applied in nuclear physics (for a review, see ref. [1]). In particular, quantum algebra $SU_q(2)$ has been used in describing the rotational spectra of deformed even-even nuclei [2] and superdeformed nuclei [3]. The basic model adopted in these studies is q-rotor, whose hamiltonian is written in terms of second order casimir of $SU_q(2)$. Using this model, improved fits are obtained for the rotational spectra. But the approach has been mainly phenomenological, where $SU_q(2)$ dynamical symmetry of the system was rather assumed. In any case, some physical significance could always be ascribed to the deformation parameter. For example, q has been related to the softness parameter of variable moment of inertia model [4].

On a different side, a few years back, Pegg and Barnett proposed a solution to the long standing phase operator problem of quantum harmonic oscillator [5]. A similar formalism also exists for the rotation angle operator [6]. However, although much work has been done related to phase operartor for electromagnetic fields [7,8], very little has been discussed regarding the rotation angle described in this formalism. A major reason for this indifference appears to be that 'angular velocity' was ill-defined in this framework [9]. Recently, two new solutions have been suggested to remove this malady [10,11]. So in our opinion, Barnett-Pegg (BP) formalism is a valid approach to the theory of rotation angle in quantum mechanics.

In this letter, we seek a theoretical basis for q-rotor model. The underlying framework for our theory is BP formalism. Very briefly, in this formalism, a complete set of orthogonal angle states is defined in $(2l+1)$-Hilbert,

$$|\theta_n\rangle = \frac{1}{\sqrt{2l+1}} \sum_{m=-l}^{l} exp(-im\theta_n)|m\rangle \tag{1}$$

where $\theta_n = \theta_0 + \frac{2\pi n}{2l+1}$ $(n = 0, 1, .., 2l)$. m is the eigenvalue of J_z, component of angular momentum along z-axis. The unitary operator

$$exp(\pm i\Phi) = exp(\pm i \sum_{n=0}^{2l} \theta_n |\theta_n\rangle\langle\theta_n|) \tag{2}$$

corresponds to the hermitian angle operator, Φ. It can be said that the above formalism has inherent q-deformed structure [10], if we identify $q = exp(-i\frac{2\pi}{2l+1})$. Then the operators $q^{J_z/\hbar}$ and $e^{i\Phi}$ satisfy

$$q^{J_z/\hbar} e^{i\Phi} = q\, e^{i\Phi} q^{J_z/\hbar} \tag{3}$$

which is the quantum plane condition in the discrete angular momentum-angle phase space [12]. These operators show cyclic displacement property

$$e^{i\Phi}|m\rangle = |m+1\rangle \tag{4}$$

$$e^{i\Phi}|l\rangle = e^{i(2l+1)\theta_0}|-l\rangle. \tag{5}$$

$e^{-i\Phi}$ is the lowering operator in this representation. Operator $q^{J_z/\hbar}$ has analogus action on $|\theta_n\rangle$ states. All this implies a periodic lattice structure, the lattice constant being related to the deformation parameter. It has been suggested [13] quite generally, that whenever there is an underlying lattice or presence of discrete lengths in the system, the related algebraic structure is a q-deformed one.

Thus we note that the algebra of operators in BP-formalism is q-deformed angular momentum algebra or briefly, $SU_q(2)$, with q as root of unity. For the purpose of q-rotor, it is essential to study the full set of generators of $SU_q(2)$. The algebra is given by the following commutation relations

$$[J_+, J_-] = [2J_z] = \frac{q^{2J_z} - q^{-2J_z}}{q - q^{-1}}, \tag{6}$$

$$[J_z, J_\pm] = \pm J_\pm \tag{7}$$

The operators act on the subspace spanned by angular momentum eigenstates $|jm\rangle$, where the representation label j is very less than l, which denotes the full space. Now for each value of j, operators $J_\pm$ act only on $|jm\rangle$ $(m = -j, -j+1, ..., j)$. They do not mix eigenstates with different values of j. Thus each complete set of $|jm\rangle$ states, serves as a basis for a $(2j+1)$-irrep for $SU_q(2)$. The action of $J_\pm$ is given by

$$J_\pm|jm\rangle = \hbar\{[j \mp m][j \pm m + 1]\}^{1/2}|jm \pm 1\rangle \tag{8}$$

$$J_z|jm\rangle = m\hbar|jm\rangle. \tag{9}$$

The second order casimir of this algebra is

$$C_2(SU_q(2)) = [J_z][J_z + 1] + J_-J_+, \tag{10}$$

with eigenvalues

$$C_2(SU_q(2))|jm\rangle = [j][j+1]|jm\rangle. \tag{11}$$

Now consider a set of states, each labelled by a different value of j, ranging from j_{min} to j_{max}, and incresing in equal steps. This is our *system*. Question is, how much large the total space with dimension $(2l+1)$ be ? BP theory requires [6] that for physical states it is finite but may be arbitrarily large, such that $l \gg j$. We try to set the lower bound for the $(2l+1)$ value. A natural way is to demand that the total hilbert space be sufficiently large to contain all the $(2j+1)$ irreps of the *system*. More precisely, we set

$$2l + 1 = \sum_{j_{min}}^{j_{max}} (2j + 1). \tag{12}$$

If the above sum is even, then $(2l+1)$ value is the next odd number. Remember that we are guessing the minimum size of the total space and in this way, maximally fix the deformation parameter from the theory. In short we suggest that *the size (scale) of the embedding space is determined by the sum total of dimensions of the contained irreps.* (See also the remark after eq. (15).

Now we turn to the q-rotor model. The hamiltonian is given by

$$H_q = \frac{1}{2I} C_2(SU_q(2)), \tag{13}$$

where I is the moment of inertia. The energy of a state labelled by quantum number j is then

$$\begin{aligned} E_j &= \frac{1}{2I}[j][j+1] \\ &= \frac{1}{2I}\frac{sin\ (\tau j) sin\ (\tau(j+1))}{sin^2\ (\tau)} \end{aligned} \tag{14}$$

where $\tau = \frac{2\pi}{2l+1}$ and $(2l+1)$ is determined from eq. (12).

We apply the above ideas to rotational spectra in deformed even-even nuclei and superdeformed nuclei. We have fitted the parameter $A = 1/2I$ in eq. (14) for some nuclei and the results are tabulated in table 1. The improvement of fits over the classical $j(j+1)$ formula is indicated by the decrease in the quantity

$$\chi^2 = \sum_{j_{min}}^{j_{max}} \{exp(j) - theo(j)\}^2. \tag{15}$$

Before discussing the results, a remark is in place. In the standard BP aprroach, after the physical quantities have been calculated, l is made to go to infinity. This is necessary, because only in the semiclassical limit of large angular momentum, the phase operator becomes continuous. However, we argue that q-rotor is discussed in angular momentum representaion and we make no attempt to determine physical quantities in terms of Φ. Thus the angle eigenvalues remain indeterminate. Secondly, the square bracket $[j]$, which generally can also take negative values for q root of unity, always remains positive here. This is because of $l \gg j$.

It is observed that in general, there is significant decrease in root mean square deviation showing good improvement of results over the ordinary rotor (for ^{238}U, the classical rms deviation is about 250). The most interesting observation is that for deformed even-even

nuclei, τ assumes values around 0.03. This was also found in [2], where τ was treated as fitting parameter. The fitted parameter $A = 1/2I$ also assumes similar close values.

In the case of superdeformed nuclei, τ assumes still smaller values. In 130 mass region, $\tau \approx 0.01$. For 150 mass region, $\tau \approx 0.004$. Both these results match with the phenomenological findings of [3]. In 190 mass region, τ is in between the above two values. Thus on the average, there is a monotonic decrease of τ in different mass regions with increase of superdeformation. Also it has been found on phenomenological basis [2,3], that for one deformation parameter fitting of q-rotor, only $q = e^{i\tau}$ and not $q = e^{\tau}$ can give improved fitting of the data. q-parameter inbuilt in the BP approach is already pure phase (modulus unity) and so it is suitable for describing the q-rotor.

Concluding, we have studied q-rotor within Barnett-Pegg approach. In the present work, a method to fix the deformation parameter *a priori* from the theory has been suggested. We have applied the idea to describe rotational spectra in deformed even-even and superdeformed nuclei. The results are quite close to those of previous phenomenological approaches to the q-rotor problem. Attempt is underway to understand in a similar vein, the rotational molecular spectra. One may look forward to describe excited collective $\beta-$ and $\gamma-$ bands of even deformed nuclei starting from the above theory. However, as pointed out in [15], alone $SU_q(2)$ symmetry may not be sufficient as these excited bands involve non-rotational degrees of freedom also. It may be possible to formulate phase theories more general than BP theory, which would incorporate deformed symmetries other than $SU_q(2)$.

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Table 1. Fitting of eq. (14) with the theoretically determined value of deformation parameter τ. Upper box is for deformed even-even nuclei while lower box shows superdeformed nuclei. Data for deformed nuclei is as used in [3,4]. For superdeformed nuclei, data is taken from ref. [14]. A is in keV and χ^2 is measured in $(MeV)^2$.

Nucleus	τ	$A = 1/2I$	$10^3\ \chi^2$
^{162}Dy	.0332	12.81	2.22
^{174}Yb	.0273	12.26	2.81
^{176}Yb	.0332	13.32	2.98
^{178}Hf	.0332	14.05	37.66
^{232}U	.0273	7.15	8.19
^{238}U	.0145	6.16	162.2
^{238}Pu	.0332	7.39	0.5
^{130}La	.0123	11.43	4.07
^{134}Nd	.0103	9.80	1.83
^{136}Nd	.0095	10.10	2.0
^{146}Gd-1	.0041	6.33	5.93
^{146}Gd-2	.0045	6.48	19.10
^{150}Tb	.0039	6.26	13.77
^{152}Dy-1	.0038	6.27	10.02
^{190}Hg	.0074	5.43	5.86
^{192}Hg	.0057	5.10	8.87
^{194}Hg-1	.0058	5.05	8.17
^{194}Hg-2	.0074	5.09	0.98
^{194}Hg-3	.0075	5.11	0.99